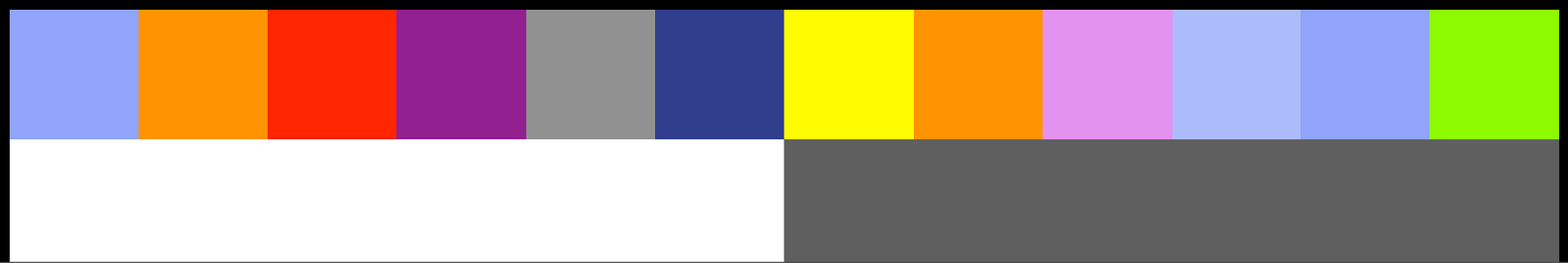

COMPUTER SECURITY LAB
RICE UNIVERSITY



Crypto and e-Voting: Homomorphisms, Zero-Knowledge Proofs

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Back to basics: Diffie-Hellman & Elgamal Crypto

Modular arithmetic 101

We're working in $\mathbb{Z}_p^*$, the integers in [1, *p*)

$$2+3 = 5 \pmod 7$$
$$2+4 = 6 \pmod 7$$
$$2+5 = 0 \pmod 7 \longleftarrow \textbf{Forbidden!}$$
$$2*3 = 6 \pmod 7$$
$$2*4 = 1 \pmod 7$$
$$6*6 = 1 \pmod 7$$

Note: $\mathbb{Z}_p^*$ is closed under multiplication but not addition.

Modular arithmetic 101

In $\mathbb{Z}_p^*$, we want to find *generators* such that

$$g^1, g^2, \ldots, g^{p-1}$$

cover all the elements in the group.

Example, for *p*=7:

g=2 is not a generator, but *g*=3 is.

Discrete logarithms

Back to the regular integers, say I give you a very big number $q = 5^{84375912432595 43}$

and ask you to take $\log_5 q$

Logarithms, over integers, are tractable. But what about in $\mathbb{Z}_p^*$**?**

No known efficient solution to DLog problem.

Diffie-Hellman (1976)

$$\begin{aligned}
\text{Alice} &: \quad \textit{random } a \in \mathbb{Z}_p^* \\
\text{Bob} &: \quad \textit{random } b \in \mathbb{Z}_p^* \\
\text{Public} &: \quad \textit{generator } g \in \mathbb{Z}_p^* \\
A \to B &: \quad g^a \\
B \to A &: \quad g^b \\
\text{Alice} &: \quad \text{computes } (g^b)^a = g^{ab} \\
\text{Bob} &: \quad \text{computes } (g^a)^b = g^{ab} \\
\text{Eve} &: \quad \text{knows } g^a, g^b, \text{cannot compute } g^{ab}
\end{aligned}$$

Elgamal encryption (1984)

Non-deterministic cryptosystem (different *r* every time)

$$
\begin{aligned}
E(g^a, r, M) &= \langle g^r, (g^a)^r M \rangle \\
D(g^r, g^{ar} M, a) &= \frac{g^{ar} M}{(g^r)^a} \\
&= M
\end{aligned}
$$

g	group generator
M	plaintext (message)
r	random (chosen at encryption time)
a	(private) decryption key
g^a	(public) encryption key

Homomorphic property

Anybody can combine two ciphertexts to get a new one.

$$
\begin{aligned}
E(M_1) \oplus E(M_2) &= < g^{r_1}, (g^a)^{r_1} M_1 > \oplus < g^{r_2}, (g^a)^{r_2} M_2 > \\
&= < g^{r_1} g^{r_2}, (g^a)^{r_1} M_1 (g^a)^{r_2} M_2 > \\
&= g^{r_1 + r_2}, g^{a(r_1 + r_2)} M_1 M_2 \\
&= E(M_1 M_2)
\end{aligned}
$$

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Homomorphic vote tallying

Change messages to counters, additive in exponent of *g*.
"Exponential Elgamal"

$$
\begin{aligned}
E(v_1) \oplus E(v_2) &= < g^{r_1}, (g^a)^{r_1} g^{v_1} > \oplus < g^{r_2}, (g^a)^{r_2} g^{v_2} > \\
&= < g^{r_1 + r_2}, g^{a(r_1 + r_2)} g^{v_1 + v_2} > \\
&= E(v_1 + v_2)
\end{aligned}
$$

g	group generator
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Violation of encryption semantics?

If I know M_1 and M_2 and $E(M_1) \oplus E(M_2) = E(M_1 M_2)$ then I can find other messages where I know their encryption!

Solution: Padding

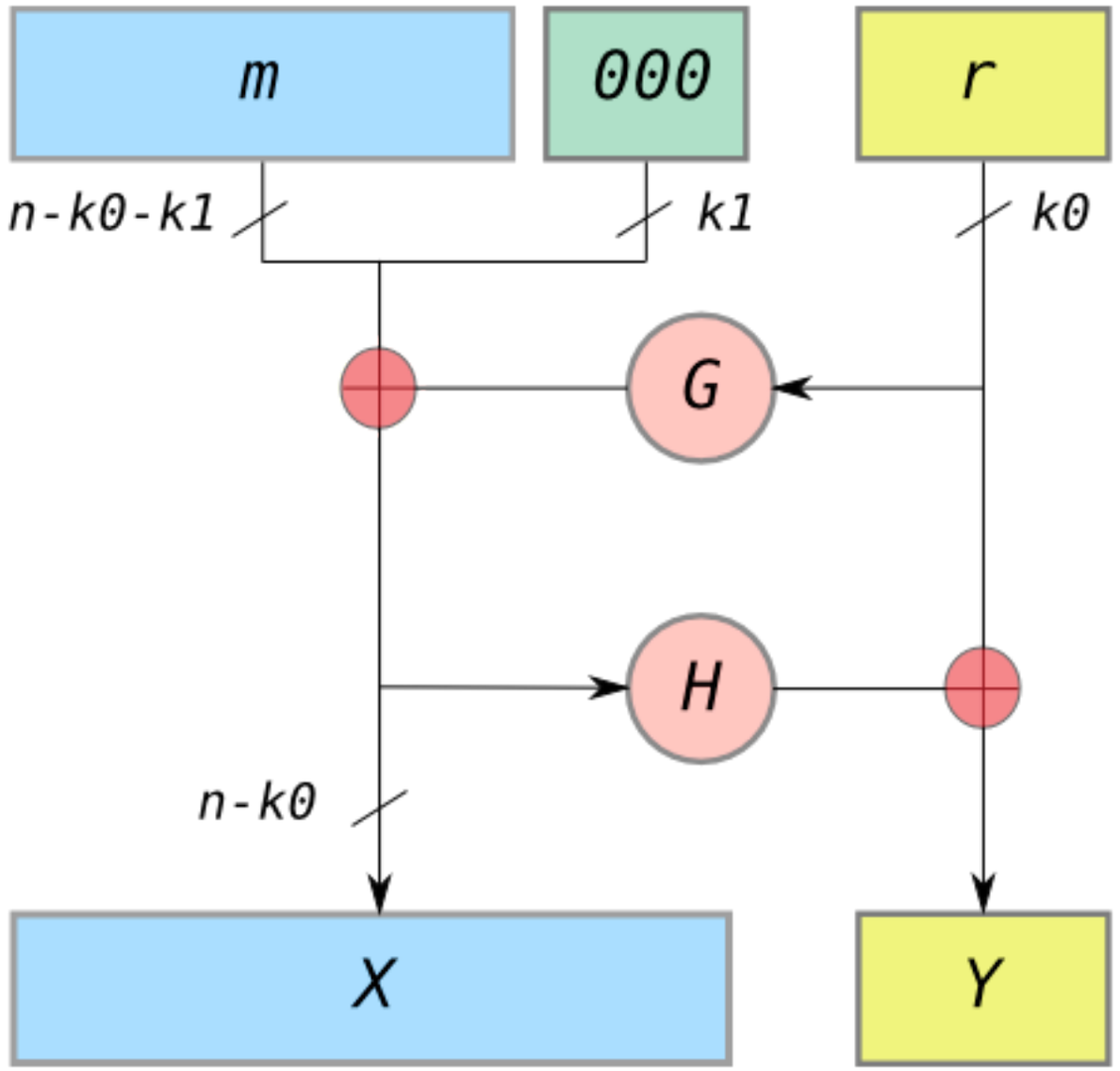
Optimal Asymmetric Encryption Padding (OAEP) - *Belare and Rogaway (1995)*

m - message (plaintext)

r - random number

G, H - cryptographic hash functions

X, Y - the message that gets encrypted



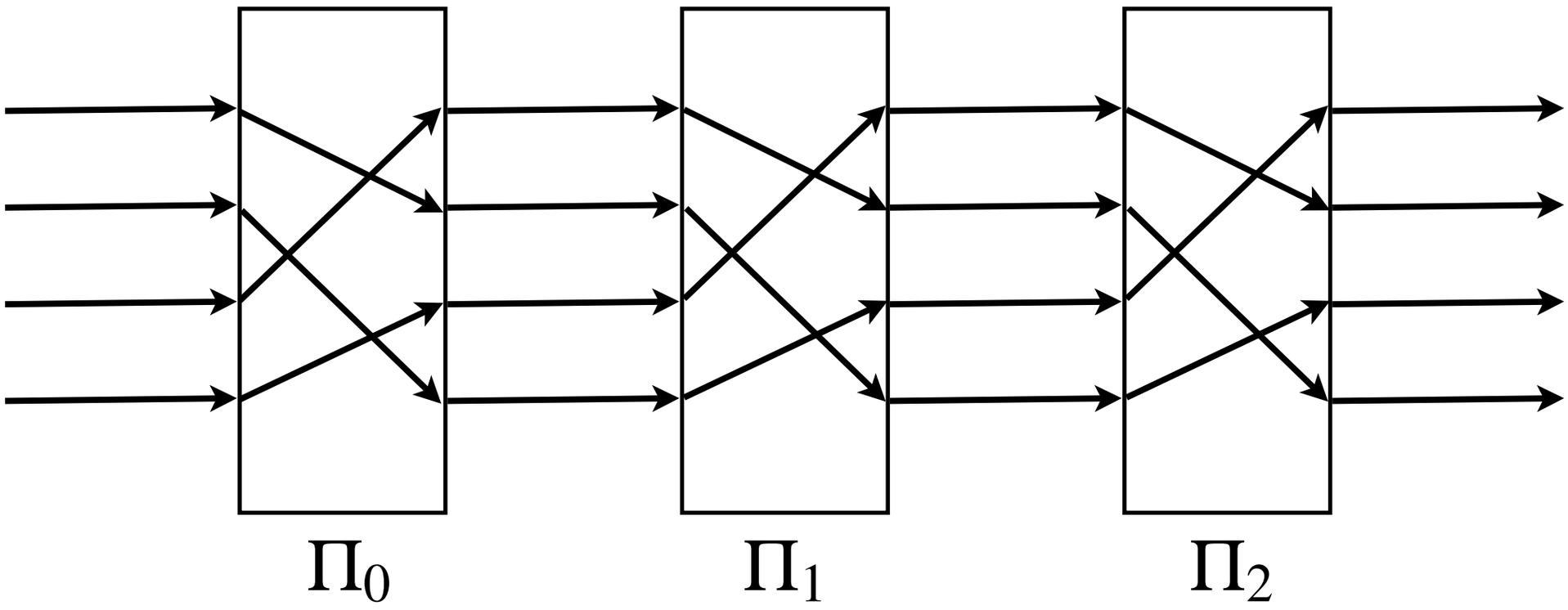
Cool trick: reencryption

$$E(M) \oplus E(0) = E(M)^*$$

Anybody can "reencrypt" a message.
(New random number introduced from $E(0)$.)

Reencryption mixnets

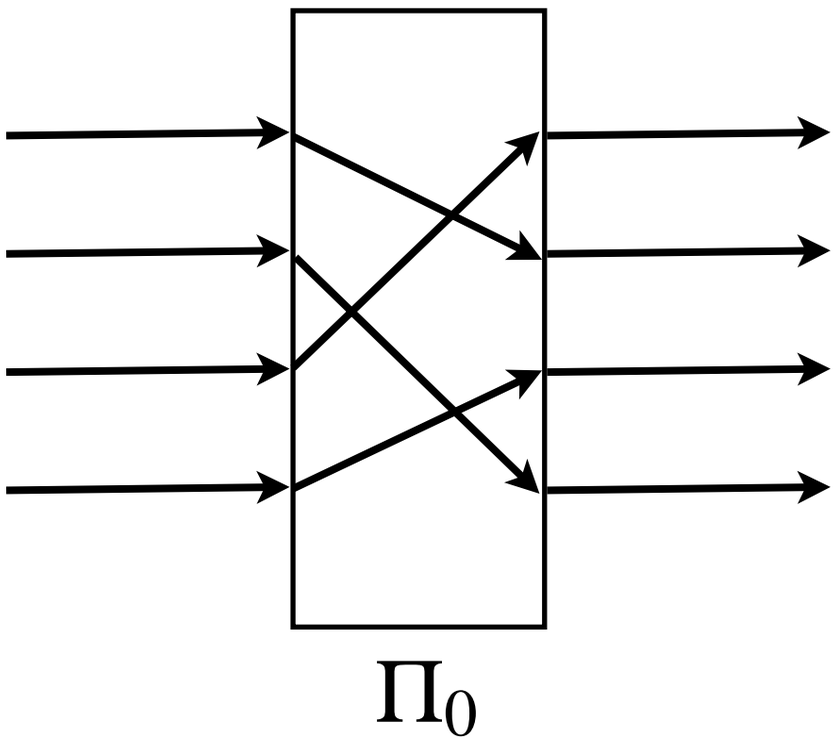
Permutations Π_i, where output is reencrypted.



Each mix permutes/reencrypts.
Must prove output corresponds to input.

Non-solution: reveal the mix

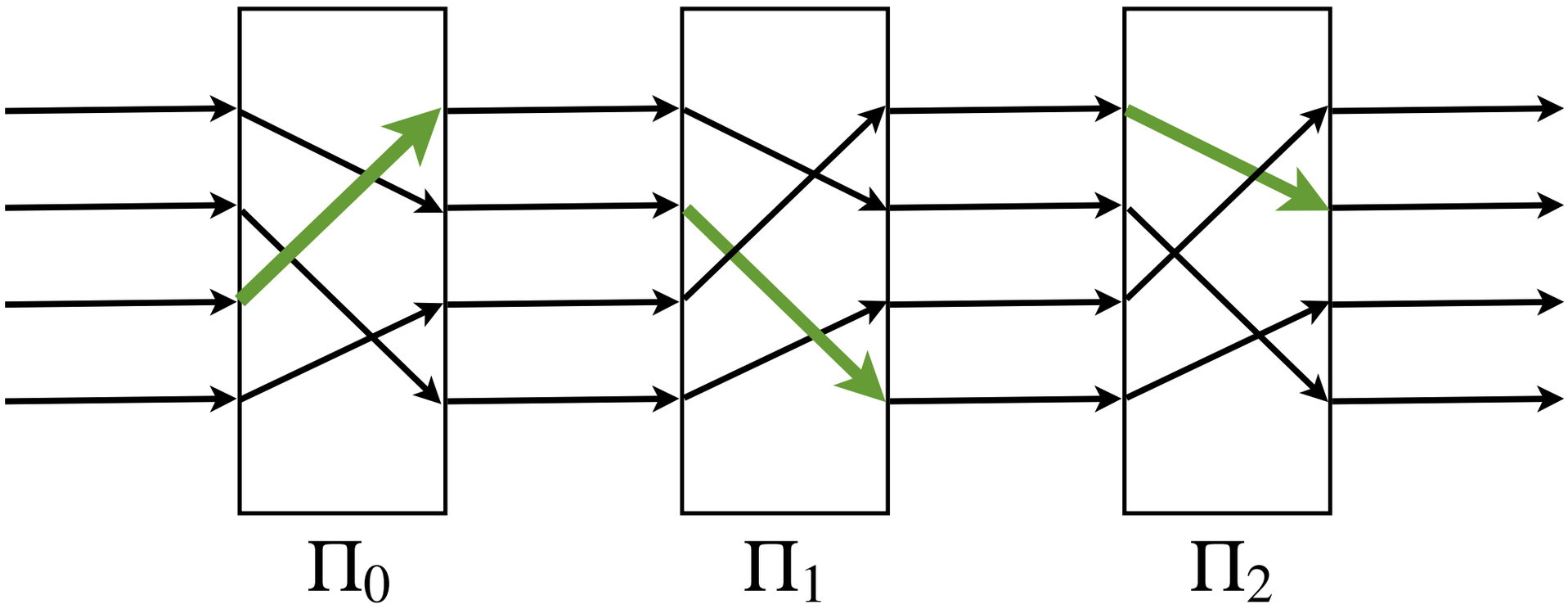
Publish the random numbers and the permutation.



Eliminates benefit of randomization.

Randomized partial checking

Effective across larger mixes.
(Jakobsson, Jules, Rivest ’02)



Say we’re mixing 1 million ballots, each mix reveals 1%. After five mixes, 99.99% chance that all ballots reencrypted at least once.

Hash functions

Widely used throughout cryptography.
Arbitrary string in.
Fixed-length string (e.g., 256 bits) out.

$$h_a = H(a)$$
$$h_b = H(b)$$

Usefulness: if $h_a \neq h_b$ then $a \neq b$.

Pre-image resistance: given h_a, it's infeasible to derive a.

Second pre-image resistance: given a and h_a, it's infeasible to derive $a' \neq a$ such that $H(a') = H(a)$

Simple use of hash functions

Tamper resistant storage: we'll store a and give you h_a

Hash chains: if new records include the hash of older records, then keep the latest somewhere safe and verify *all* older records.

Commitments: in a game like rock-paper-scissors, both parties first send the hash of their move, then reveal the moves later. (But be sure to include a random number!)

Zero-knowledge proofs (ZKP)

want to prove you know something

while revealing nothing

generalized format

prover: commit to something (e.g., reencryption mix output)

verifier: *challenge* the prover

prover: respond to the challenge

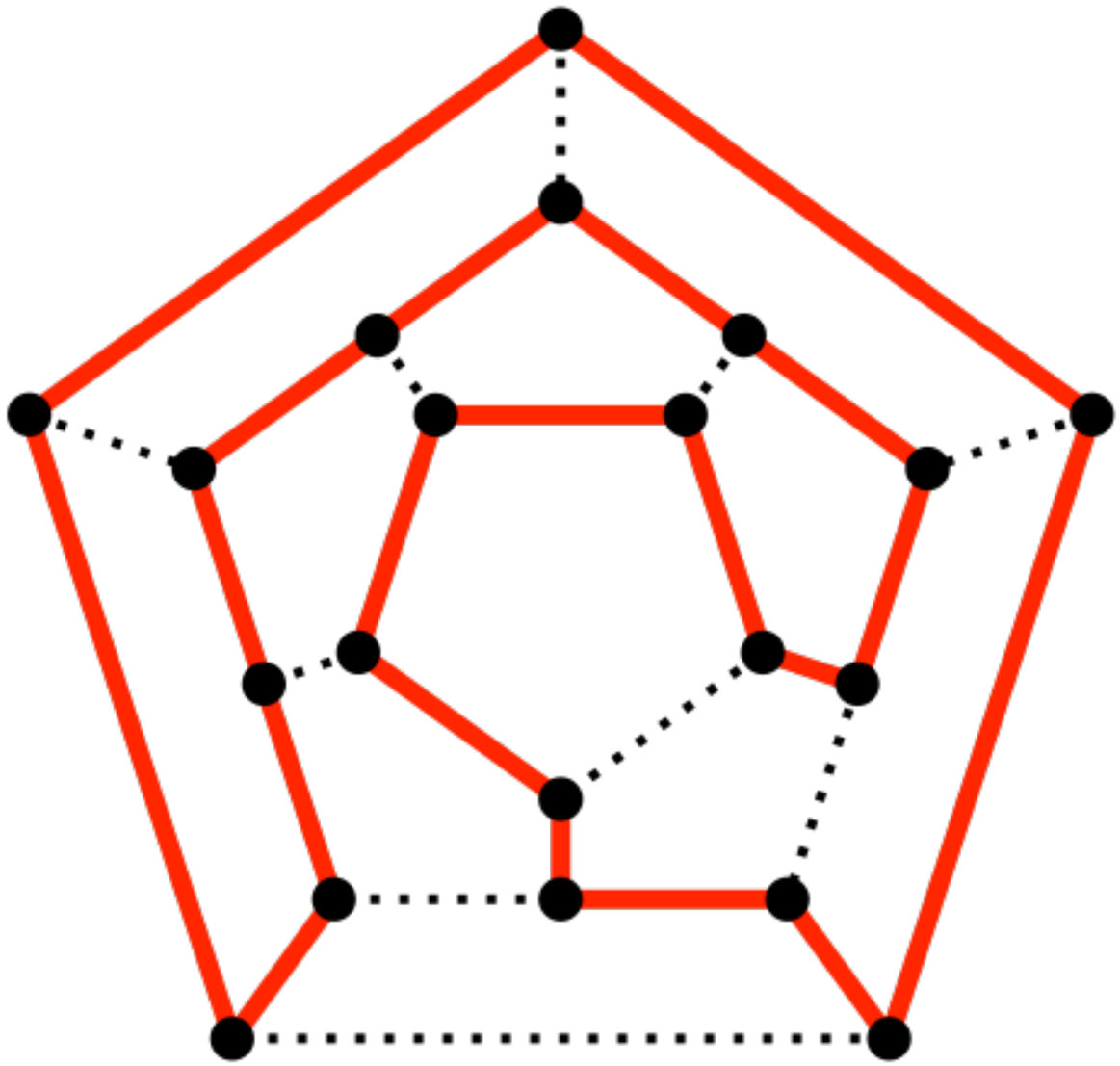
Example: Hamiltonian paths

Prover: "I know a HP over graph *G*." Compute graph isomorphism *H*. Publish *G, H.*

Verifier: Coin toss. Heads: tell me HP over *H*. Tails: tell me isomorphism *G* to *H*.

(Repeat *N* times.)

If prover doesn't know HP, verifier catches with high probability.

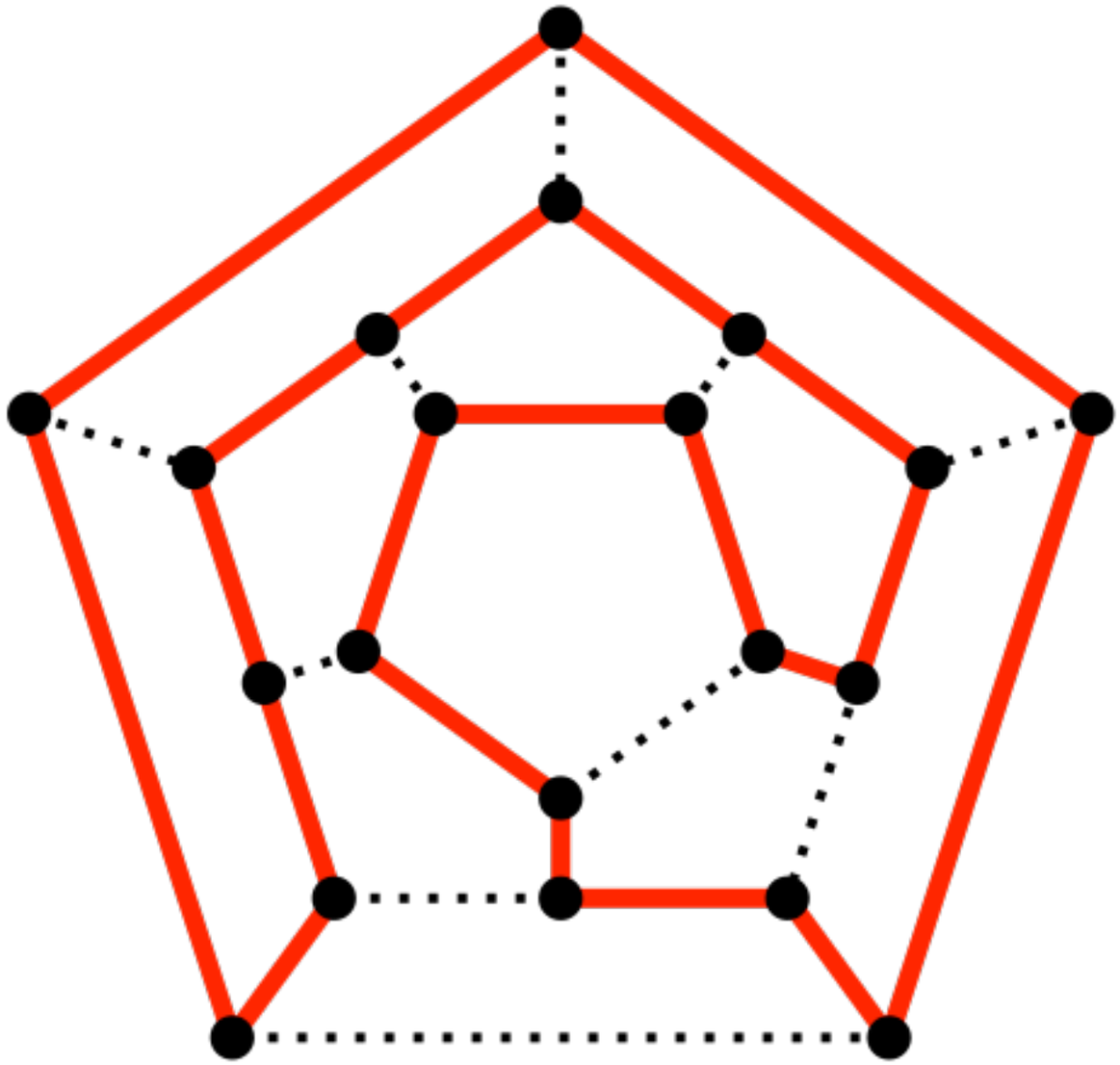


Non-interactive ZK proofs

Prover: Precompute *N* isomorphisms (H_1 to H_N) and hash them. Hash function yields coin tosses for virtual challenger. Then output the results.

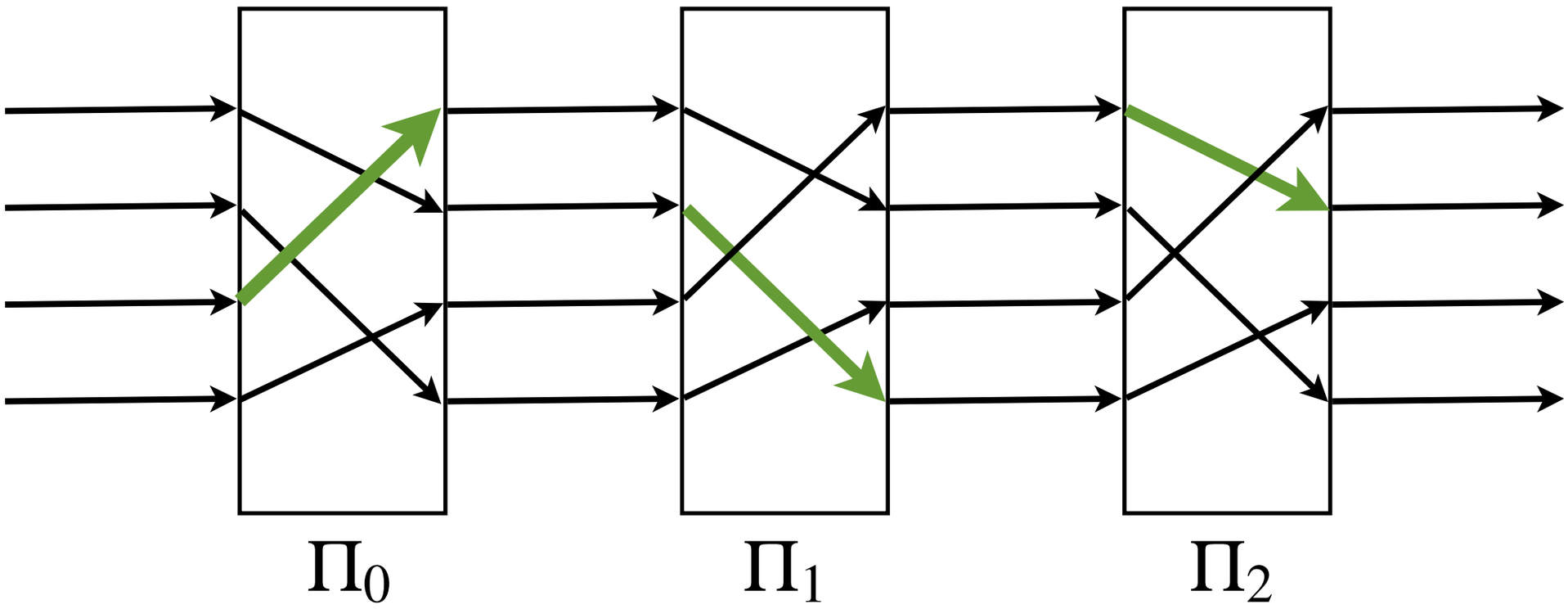
(Assumes good hash functions.)

This is an example of the *Fiat-Shamir heuristic* (1986).



NIZK variant for mixes

Hash the output of the permutation/reencryption. Use those bits to select which edges get revealed.



Say we're mixing 1 million ballots, each mix reveals 1%. After five mixes, 99.99% chance that all ballots reencrypted at least once.

Evil machine: E(bignum)?

Must prove ciphertext corresponds to well-formed plaintext. (Example, prove counters are zero or one.)

We need another ZK tool: Chaum-Pedersen proofs.

Prover knows: $(g, g^x), (h, h^x)$

Wants to prove that these two tuples share x

Chaum-Pedersen proofs (1992)

Goal: demonstrate $(g, g^x), (h, h^x)$

P: choose random $w \in \mathbb{Z}_p^*$, compute $(A = g^w, B = h^w)$

Send (A,B) to V

V: pick a random number c (challenge), send to P

P: compute $R = w + xc$

send R to V

V: Compute

$$\begin{aligned} A(g^x)^c &= g^w g^{xc} \\ &= g^{w+xc} \\ &= g^R \end{aligned}$$

$$\begin{aligned} B(h^x)^c &= h^w h^{xc} \\ &= h^{w+xc} \\ &= h^R \end{aligned}$$

Fake C-P proofs?

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Fake C-P proofs?

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Send (A,B) to ***V***

P choses fake c, R s.t. $A = g^R (g^{xc})^{-1}$.

V: pick a random number c (challenge), send to P

P: compute ~~$R = w + xc$~~ R

send R to V

V: Compute $A(g^x)^c$

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Observer can compute $A(g^x)^c$...

V: Compute

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ZK protocols only work when "live" (or use Fiat-Shamir heuristic for non-interactive)

C-P for vote testing

Can I prove a vote is zero or one? First, how about proving it's zero using C-P.

Want to verify $\langle g^r, g^{ar} g^v \rangle$ for a specific value of v?

Do C-P protocol where $(g, g^x), (h, h^x)$ becomes

$$(g, g^r), \left(g^a, \frac{g^{ar} g^v}{g^v} \right)$$

We could do this for any value of v

Challenge is to do $v = 0$ and $v = 1$ at the same time.

Cramer-Damgård-Schoenmakers (1996)

Can run two Chaum-Pedersen (or any two ZK proofs like this) simultaneously, one “real” and one “simulated”.

First, fake a proof (e.g., for $v = 1$) in advance.

Then, announce the first message for both protocols. Challenger sends c, prover announced a split c_0, c_1 where $c_0 + c_1 = c$, then executes both ZK protocols.

Verifier cannot tell which one was real vs. simulated, but knows that **one** of them was real.

Crypto summary

At the end of the day, **any** election observer can now:

- verify every single ballot for being “well-formed”
  (valid Elgamal tuple, encrypted zero-or-one, etc.)
- add together all the ballots (homomorphically)
- verify a proof of the tally (Chaum-Pedersen again)
  (only the election authority can generate this)

But we have no idea if the original ciphertext corresponded to the **intent of the voter** (versus evil machine flipping votes).

One newer, useful trick:

ballot challenge

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a technique due to [Benaloh 2007]

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at the end, instead of casting your ballot:

force the machine to **show it to you**

ballot challenge

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at the end, instead of casting your ballot:

force the machine to **show it to you**

this happens on election day

no artificial testing conditions (viz., “L&A tests”)

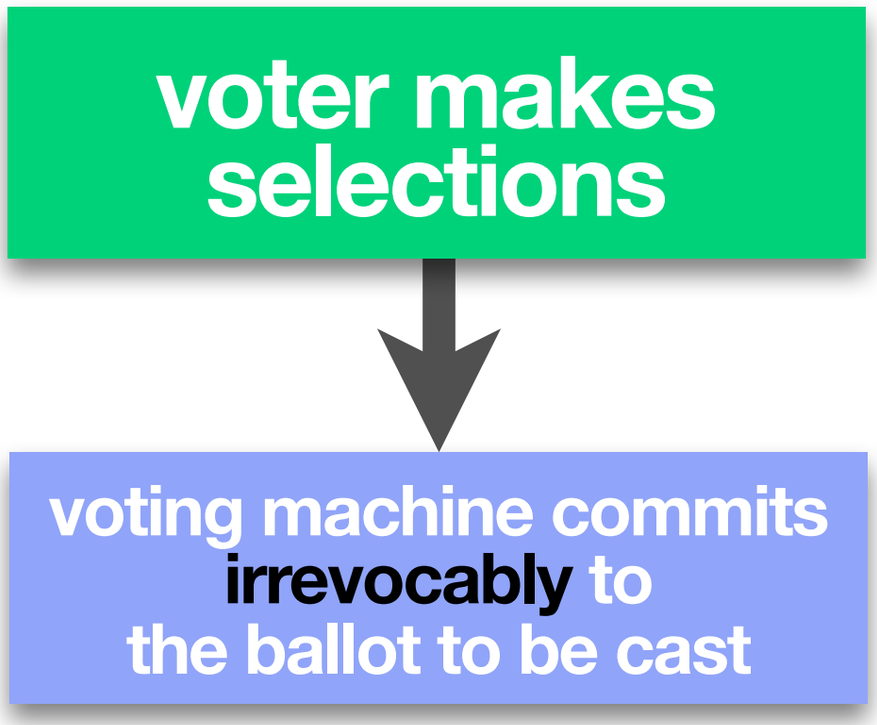
the voting machine cannot distinguish this from a real vote until the challenge

ballot challenge

ballot challenge

voter makes selections

ballot challenge



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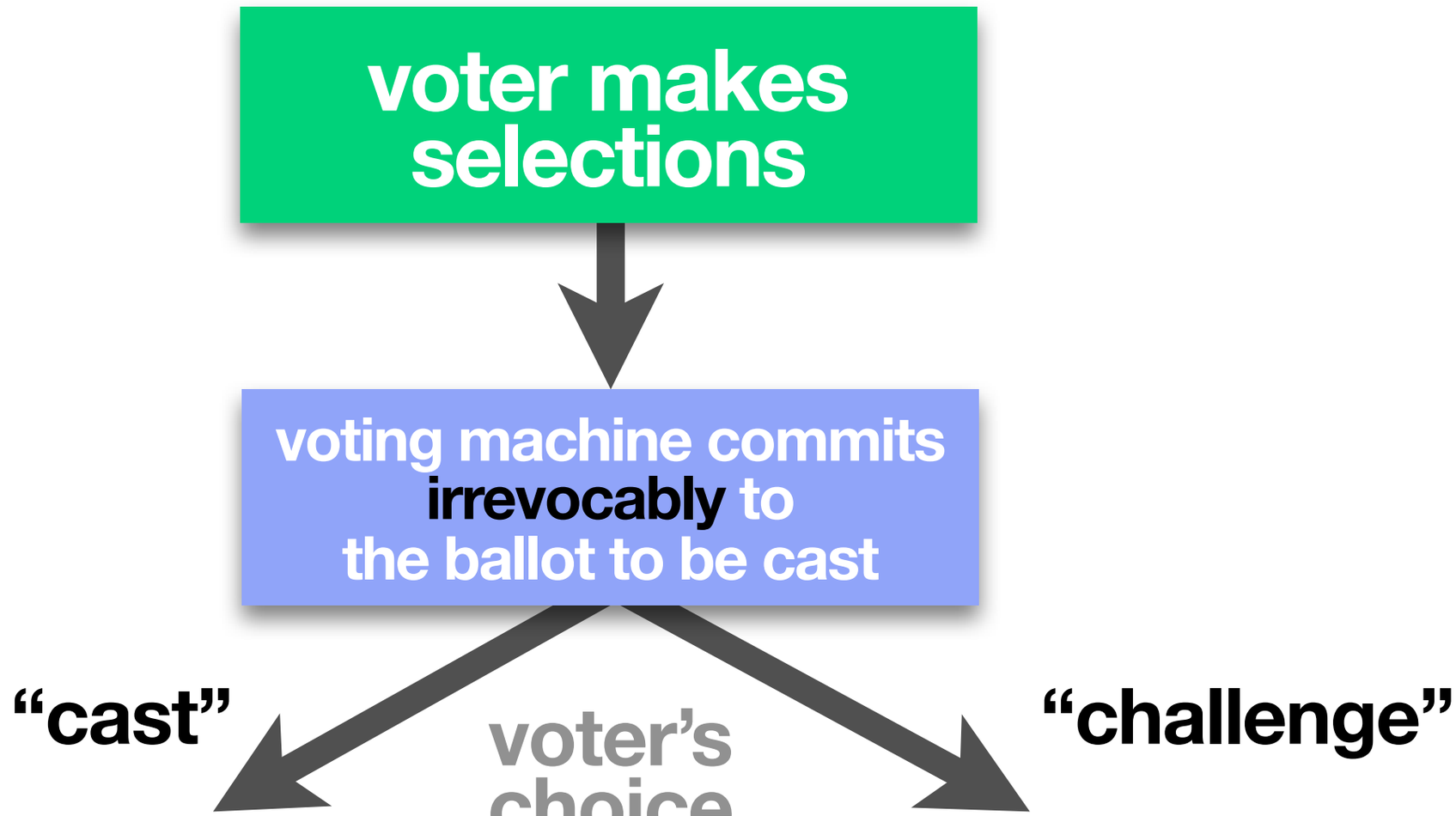

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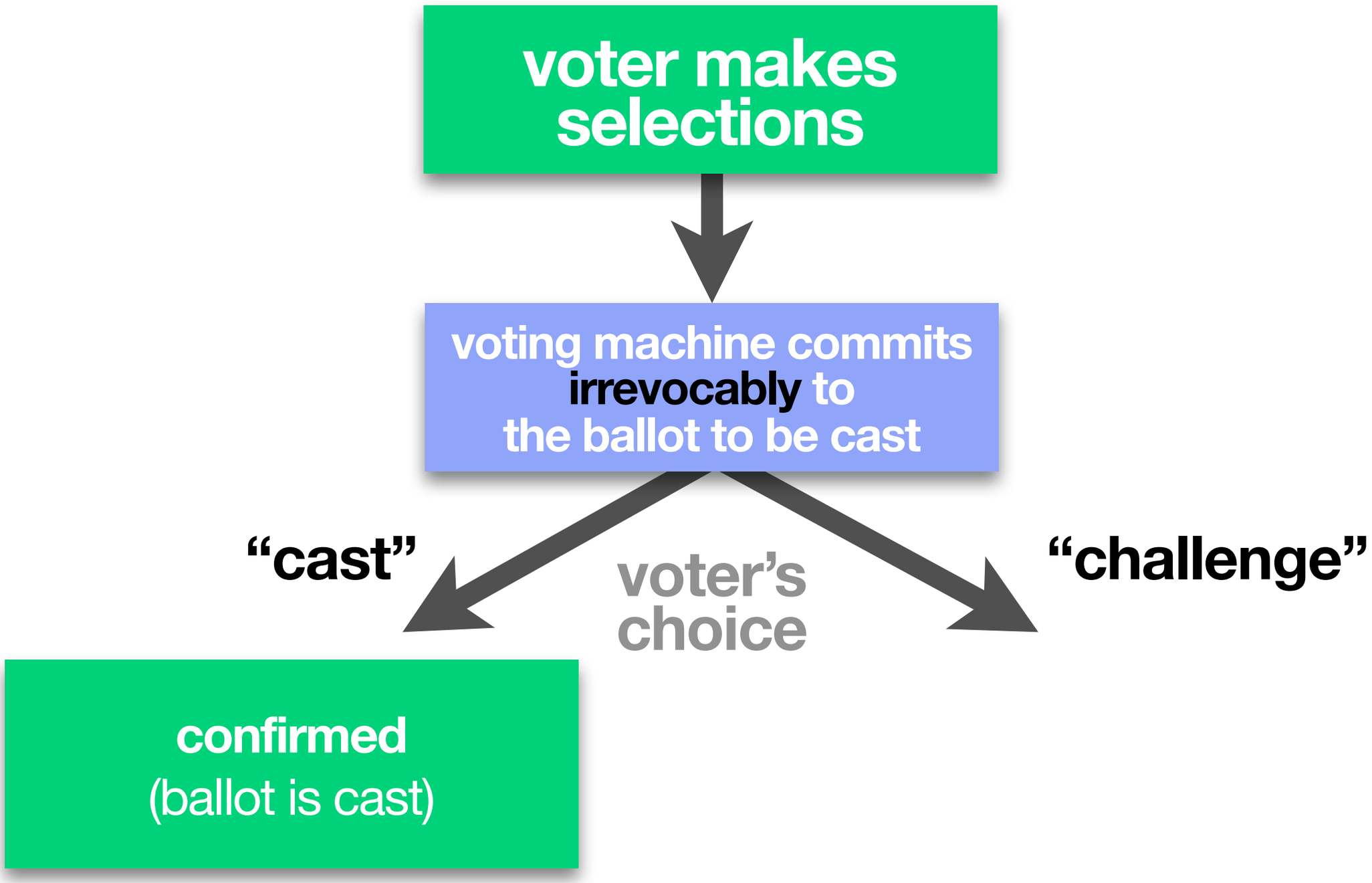

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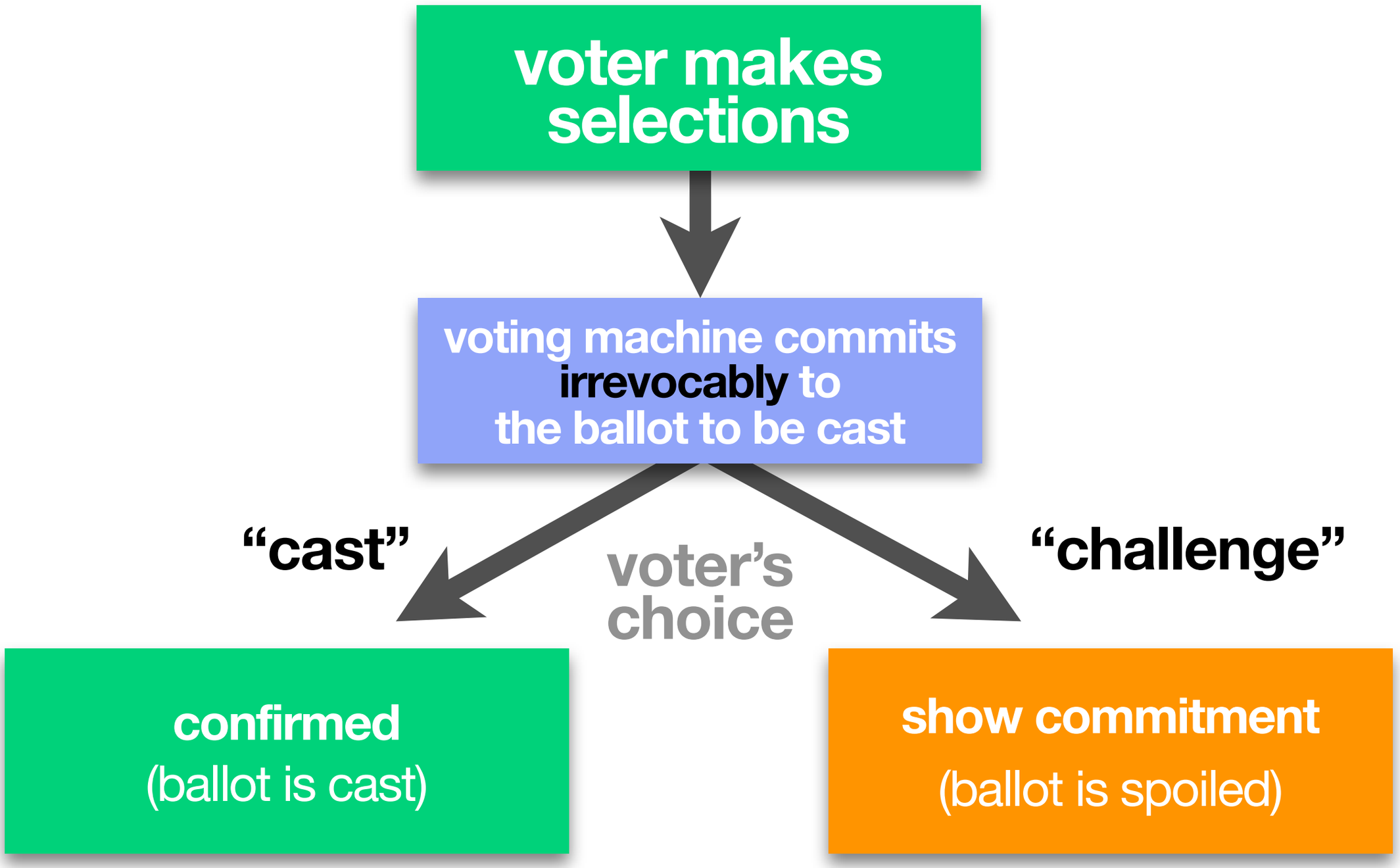

ballot commitment

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What is the commitment?

How do we force the machine to produce proof of what it's about to cast on the voter's behalf?

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Benaloh's proposal

print the encrypted ballot behind an opaque shield

You can't see the contents, but you can see the page

the computer cannot "un-print" the ballot

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How do you test the commitment?

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How do you test the commitment?

Decrypt it.

But decryption requires the private key for tabulating the whole election!

Elgamal reminder

Two ways to decrypt:

$$E(g^a, r, M) = \langle g^r, (g^a)^r M \rangle$$

$$D(g^r, g^{ar} M, a) = \frac{g^{ar} M}{(g^r)^a}$$

$$D(g^r, g^{ar} M, r) = \frac{g^{ar} M}{(g^a)^r}$$

g	group generator
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challenging the machine

challenging the machine

When challenged, the machine must reveal *r*

We can then decrypt this ballot (only) and see if it's what we expected to see

In Benaloh, the encrypted ballot is on paper

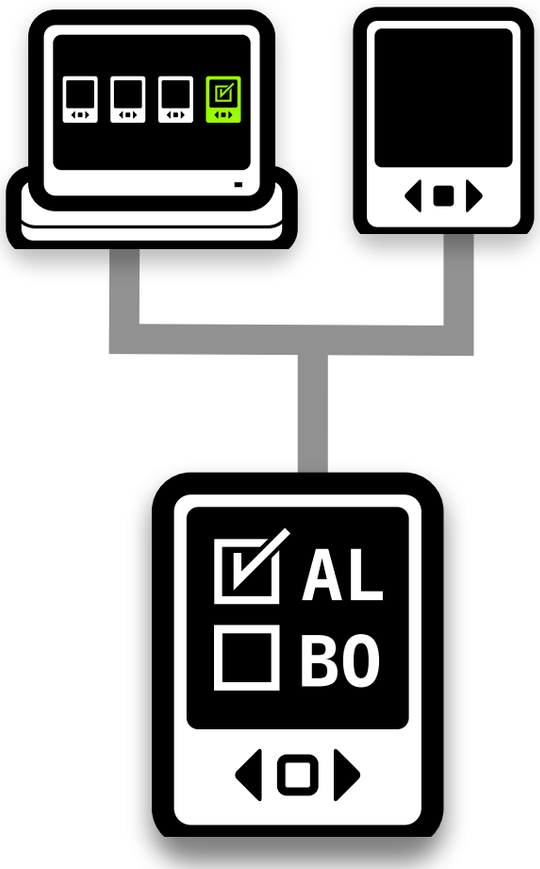
An **irrevocable** output medium

decrypting requires additional equipment

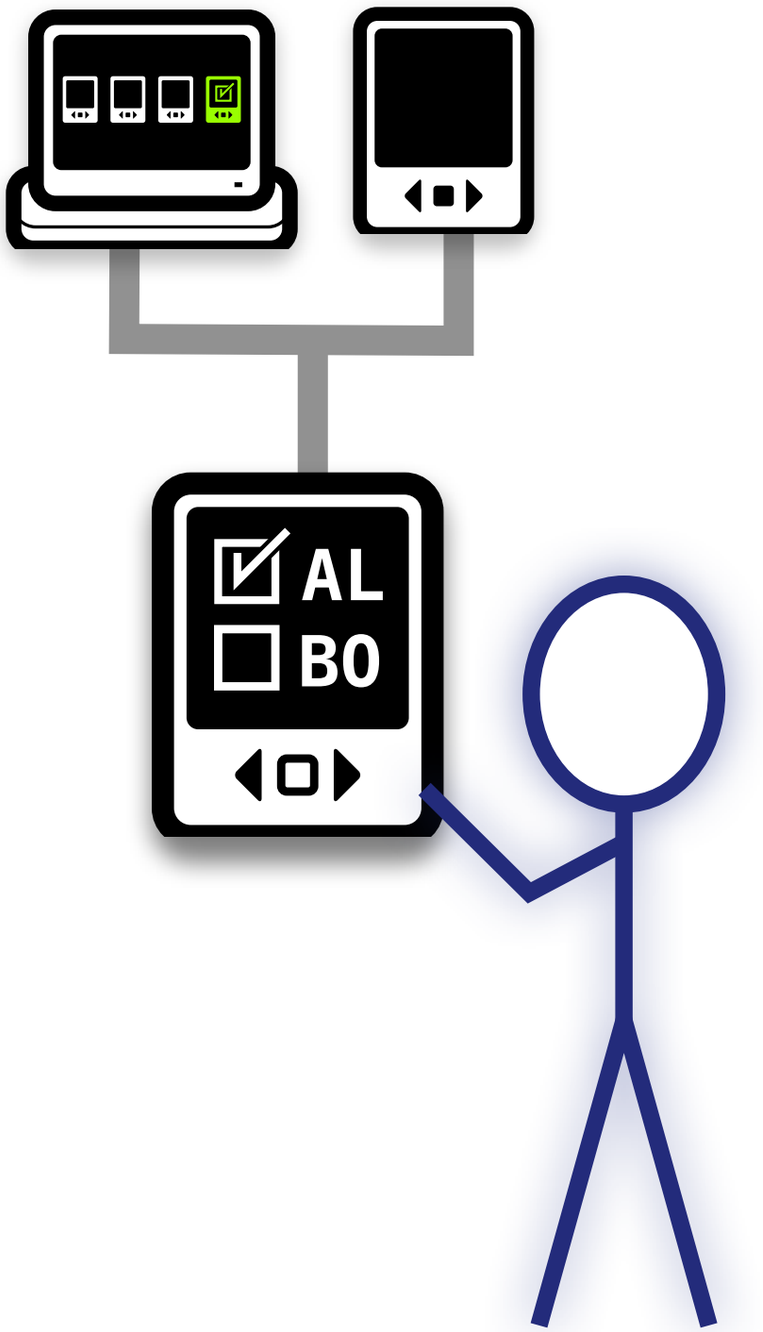
VoteBox happens to have its own irrevocable publishing system

(Its in-precinct LAN, where all machines replicate everywhere.)

polling place

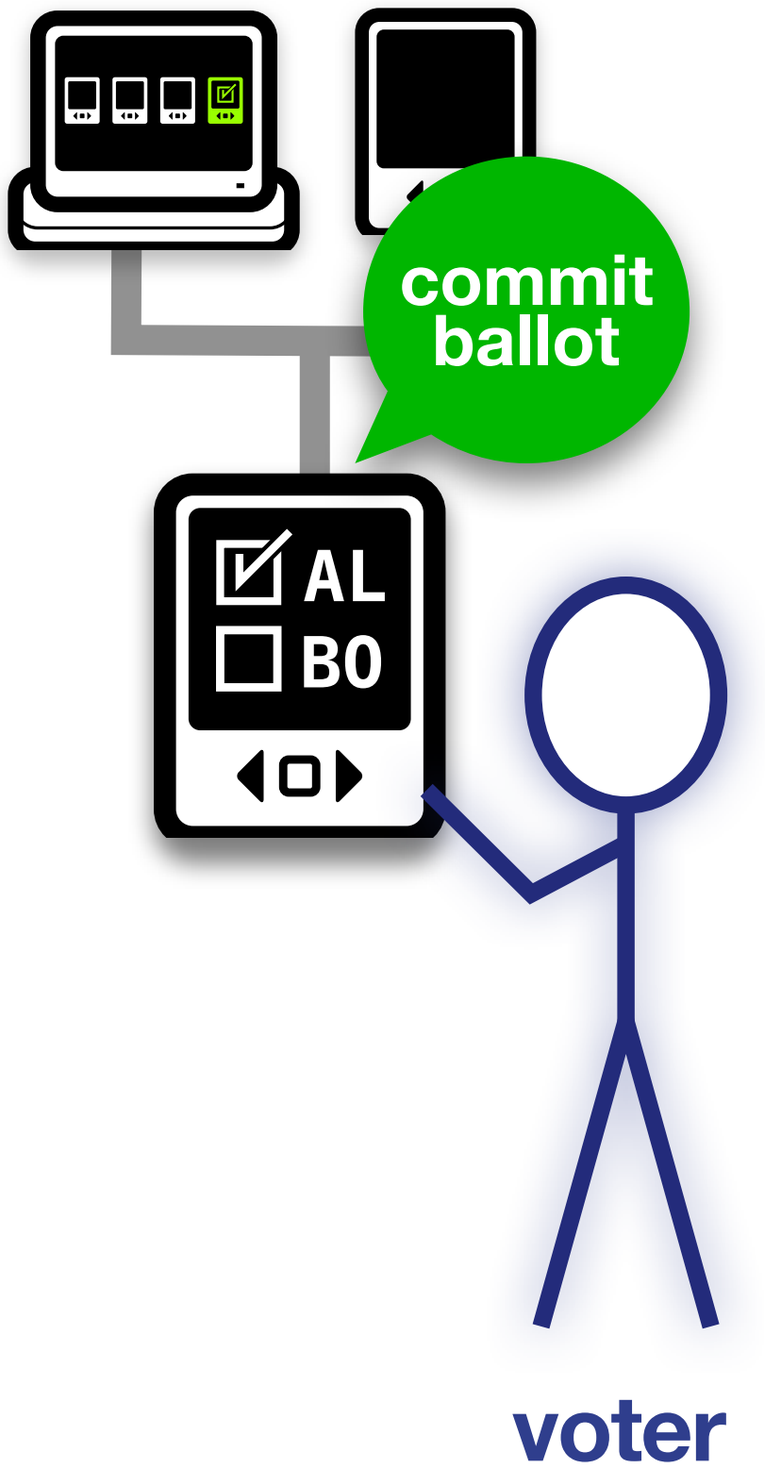


polling place



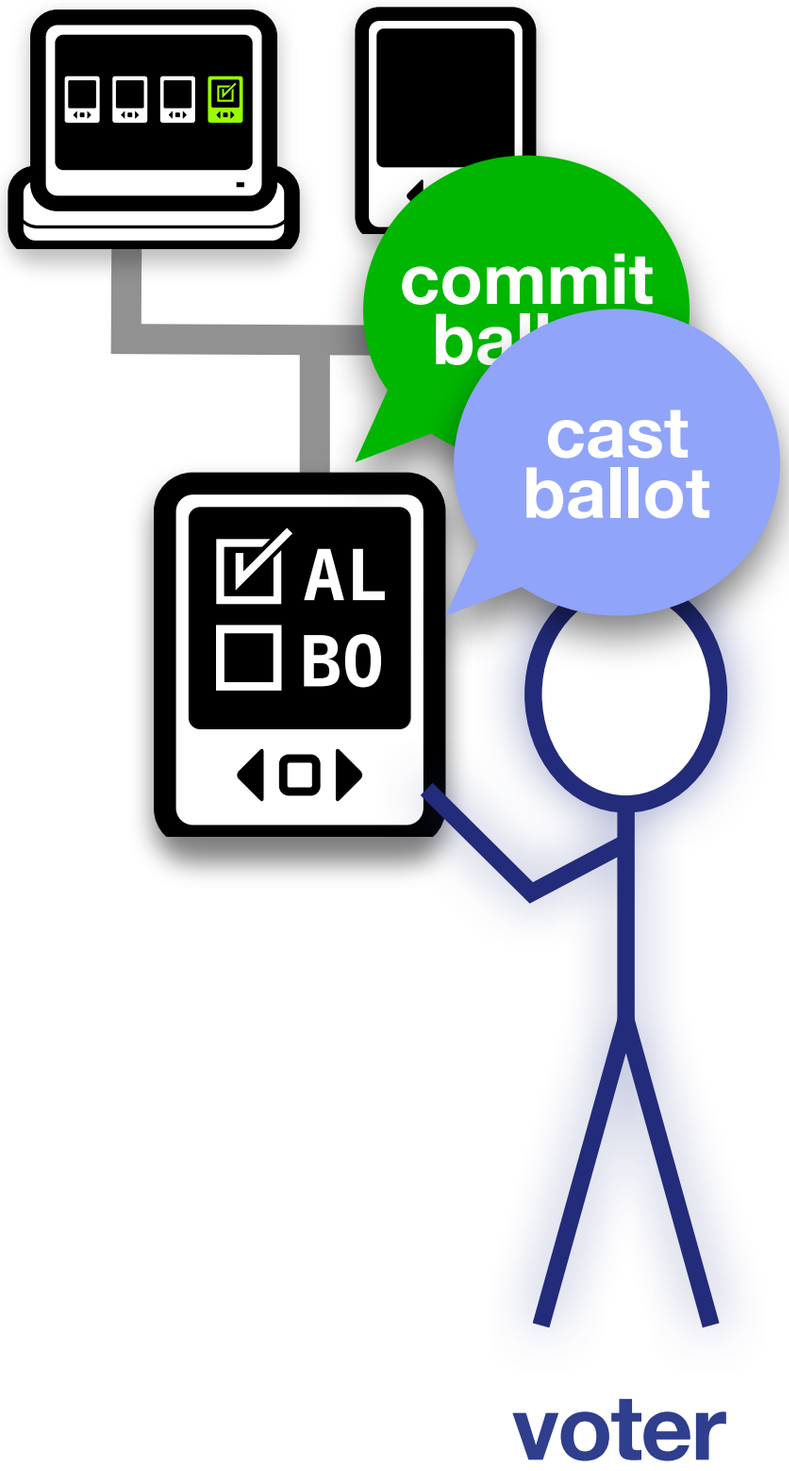
voter

polling place



voter

polling place



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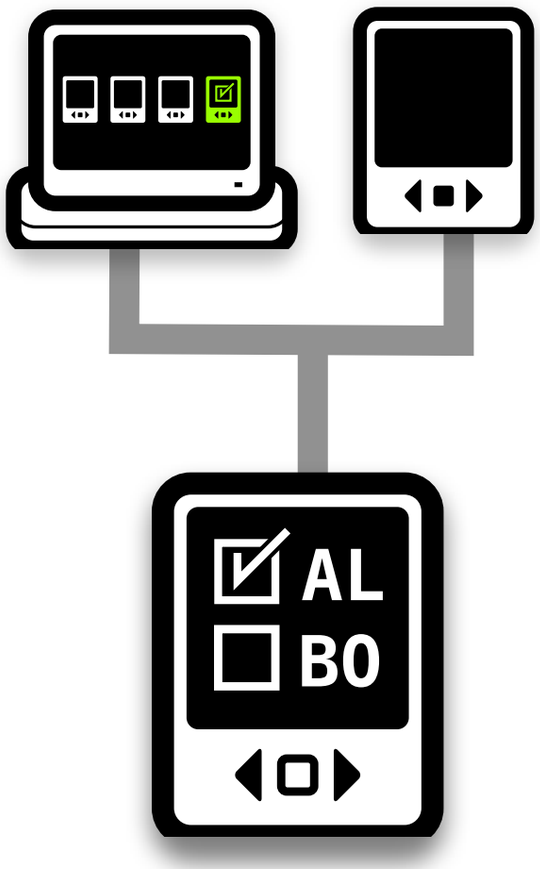

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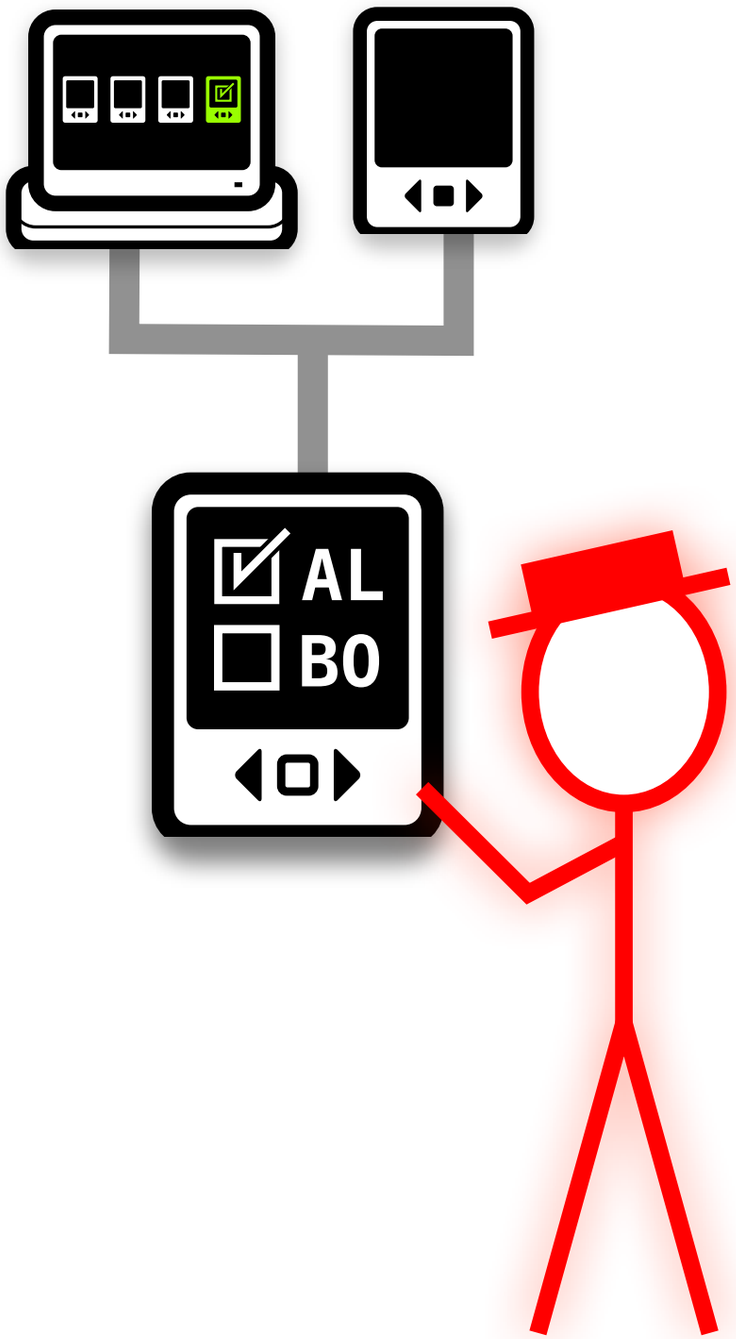
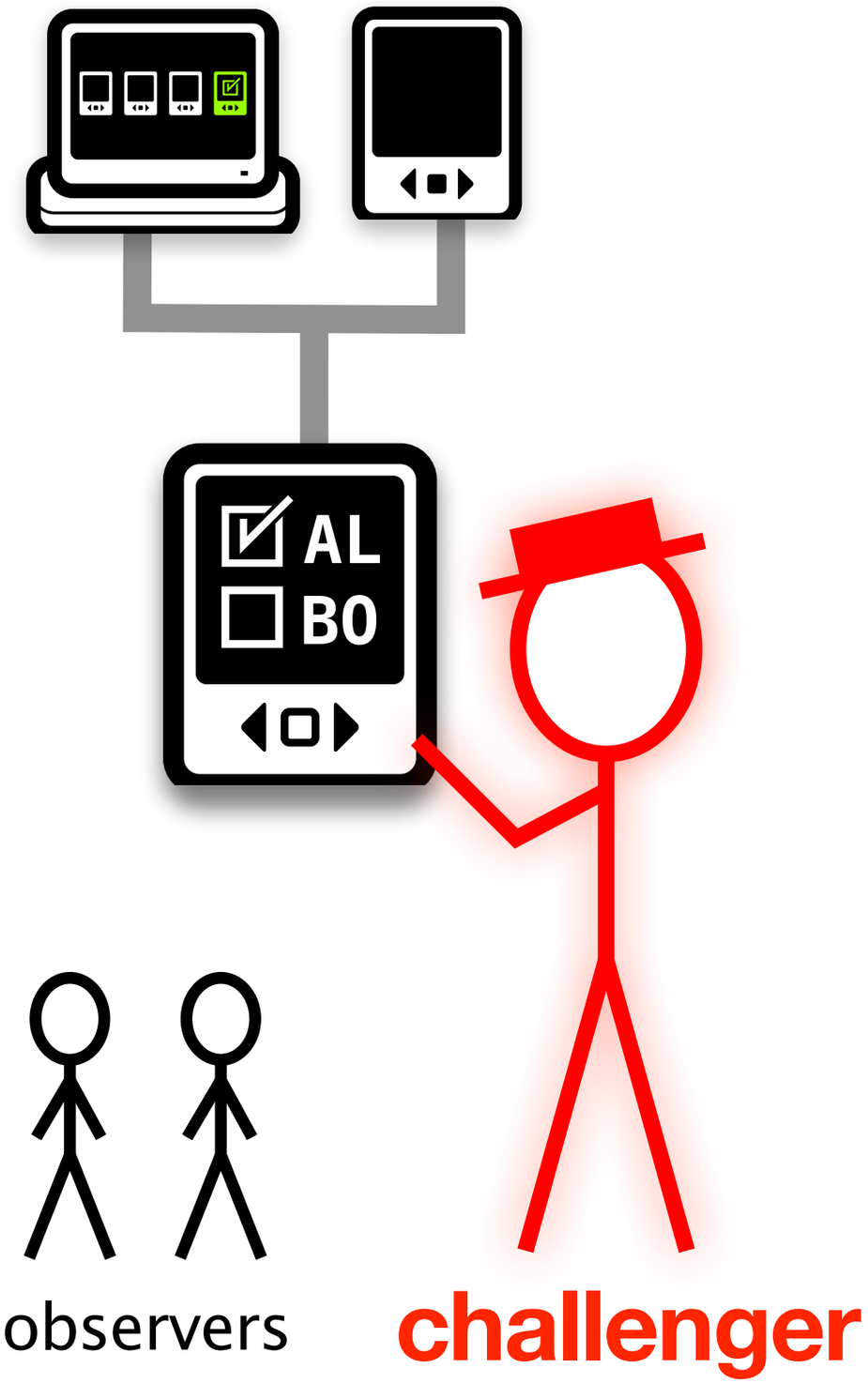


challenger

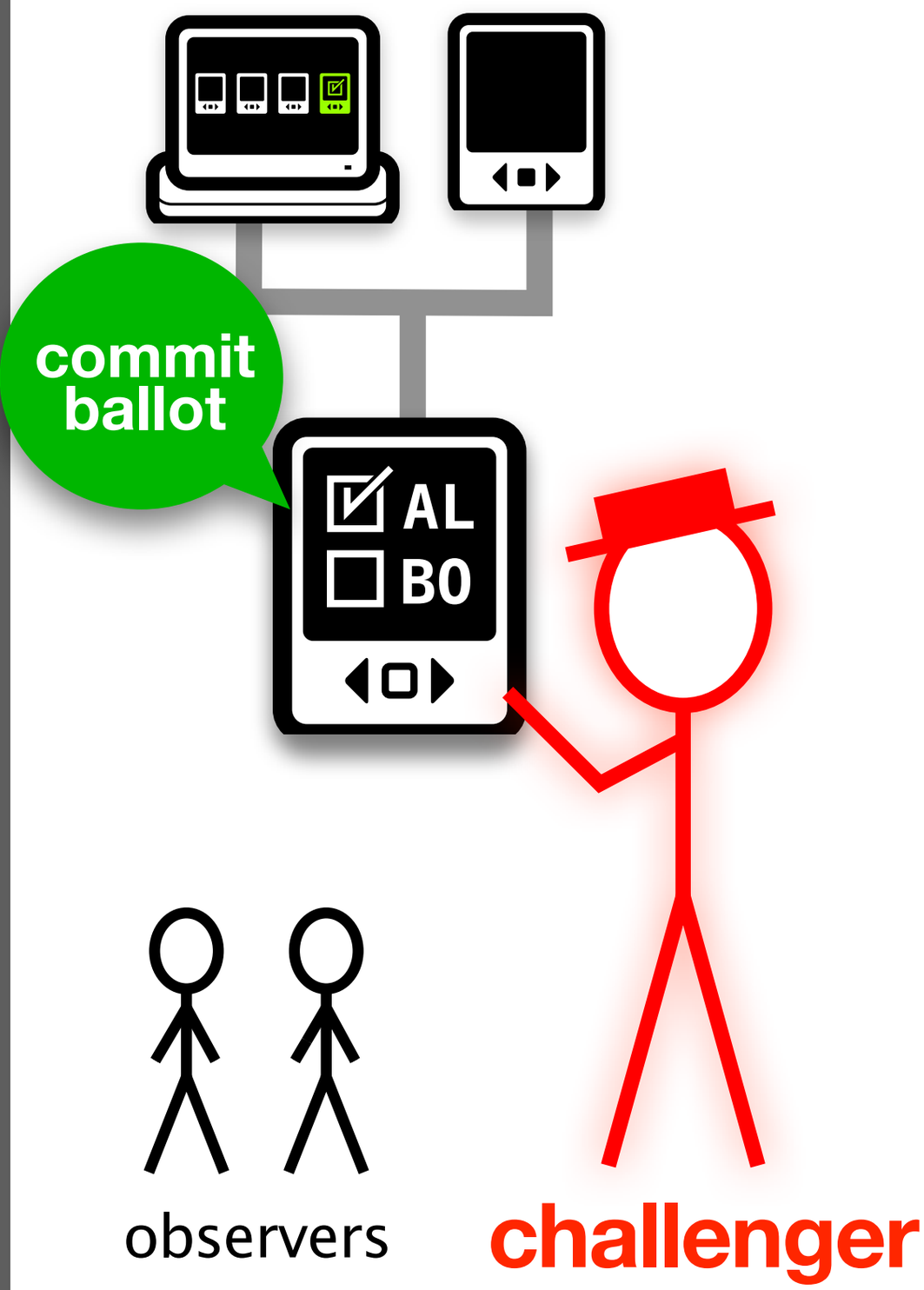
polling place



observers

challenger

polling place



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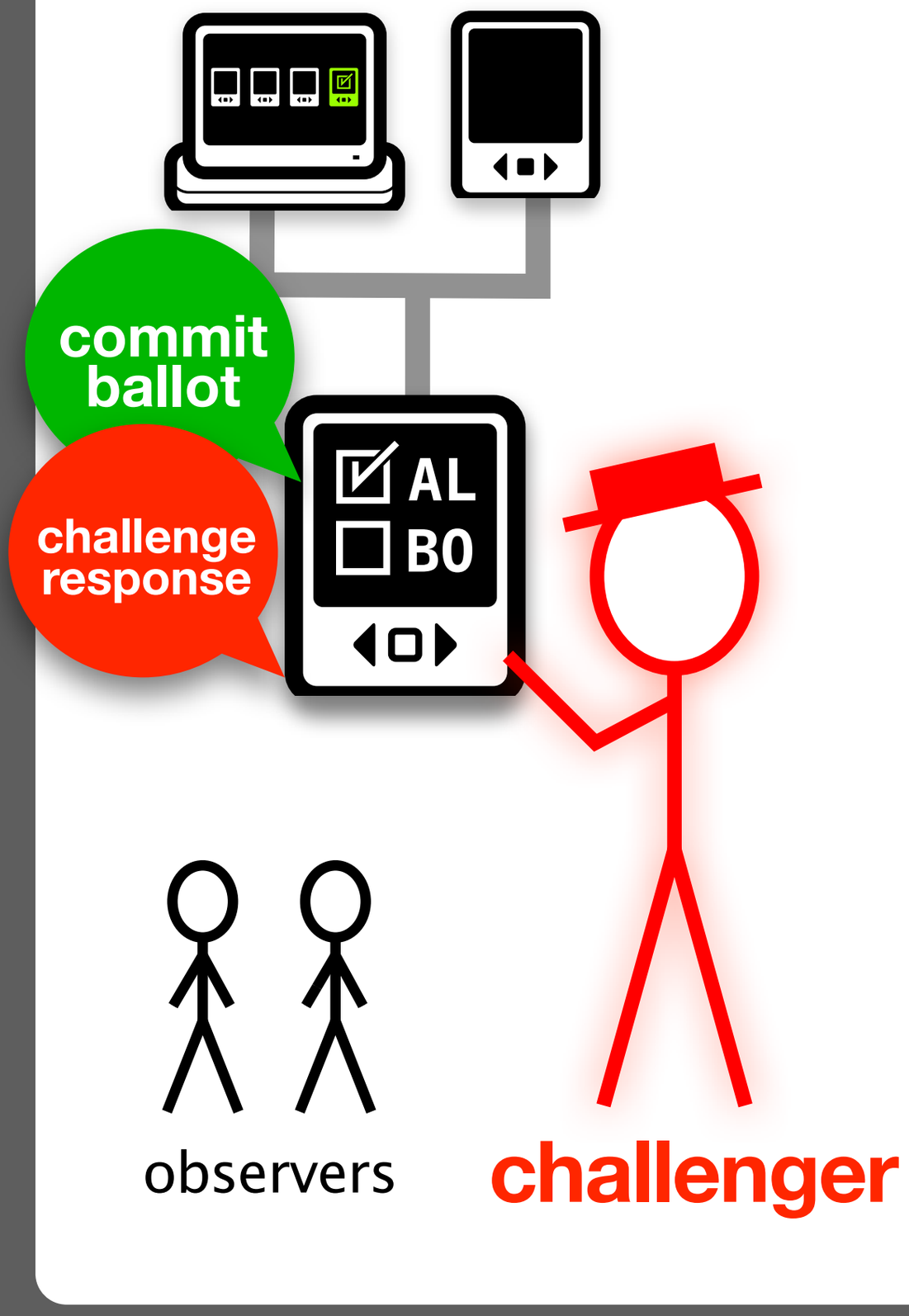

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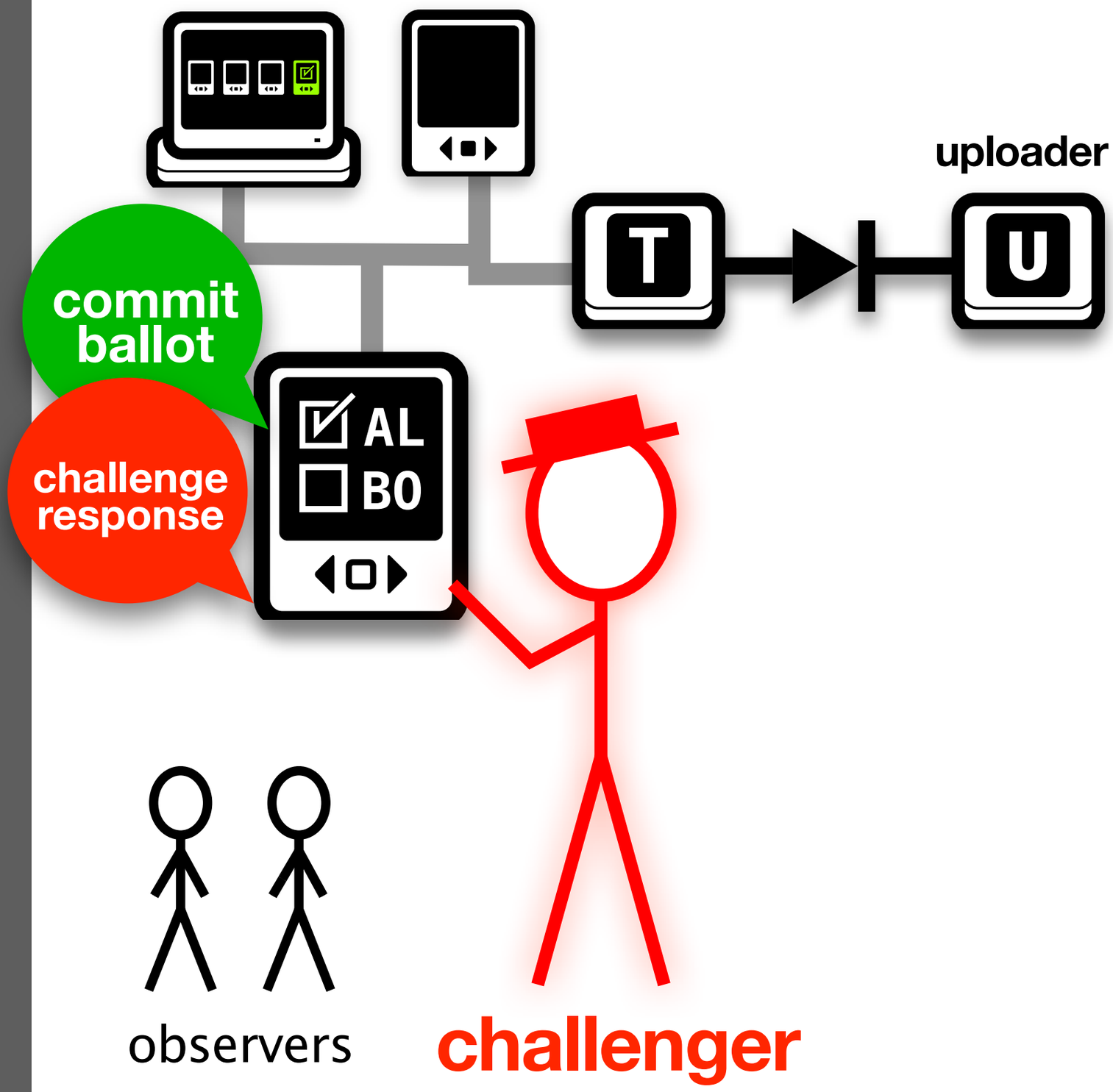

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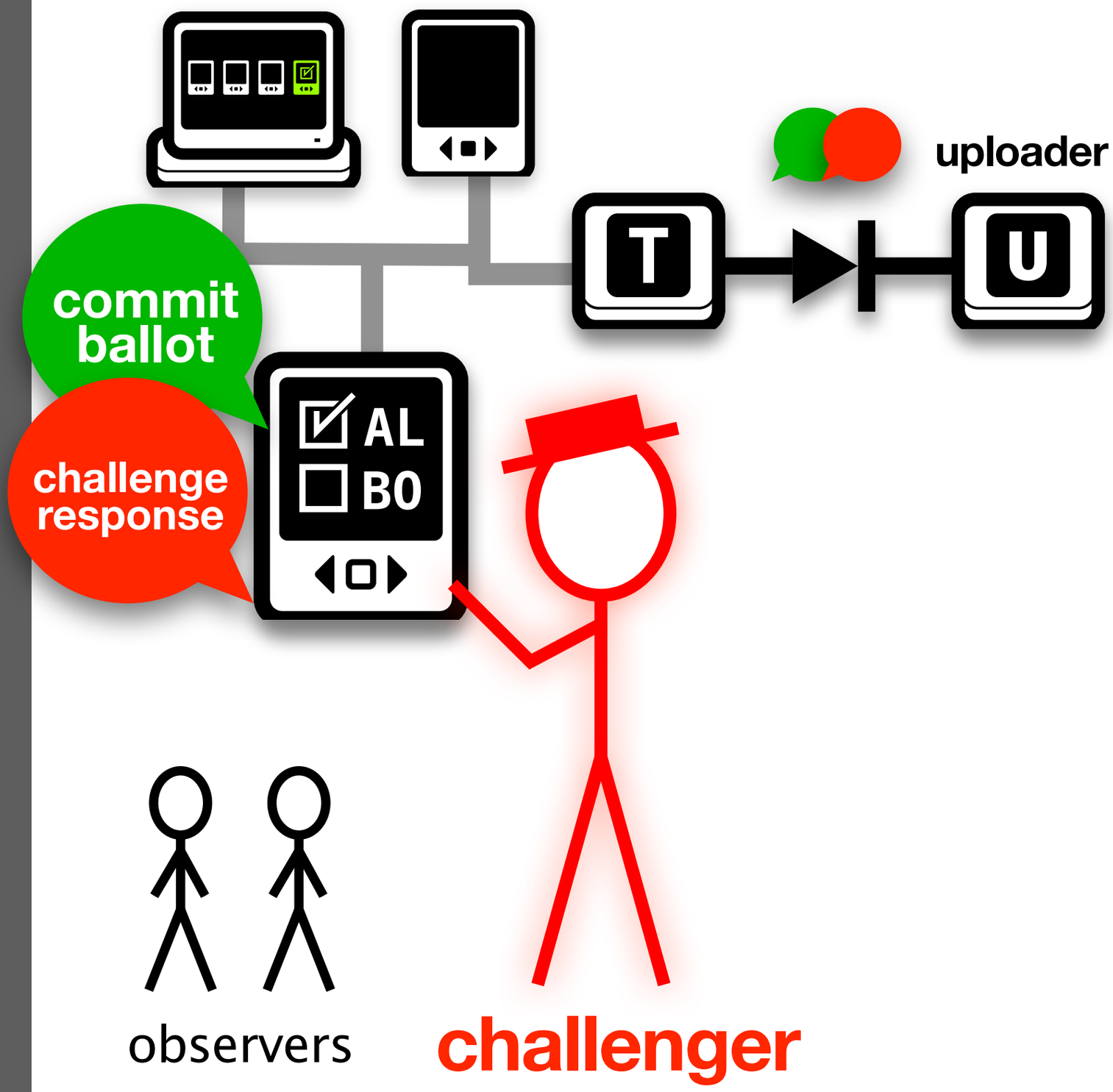
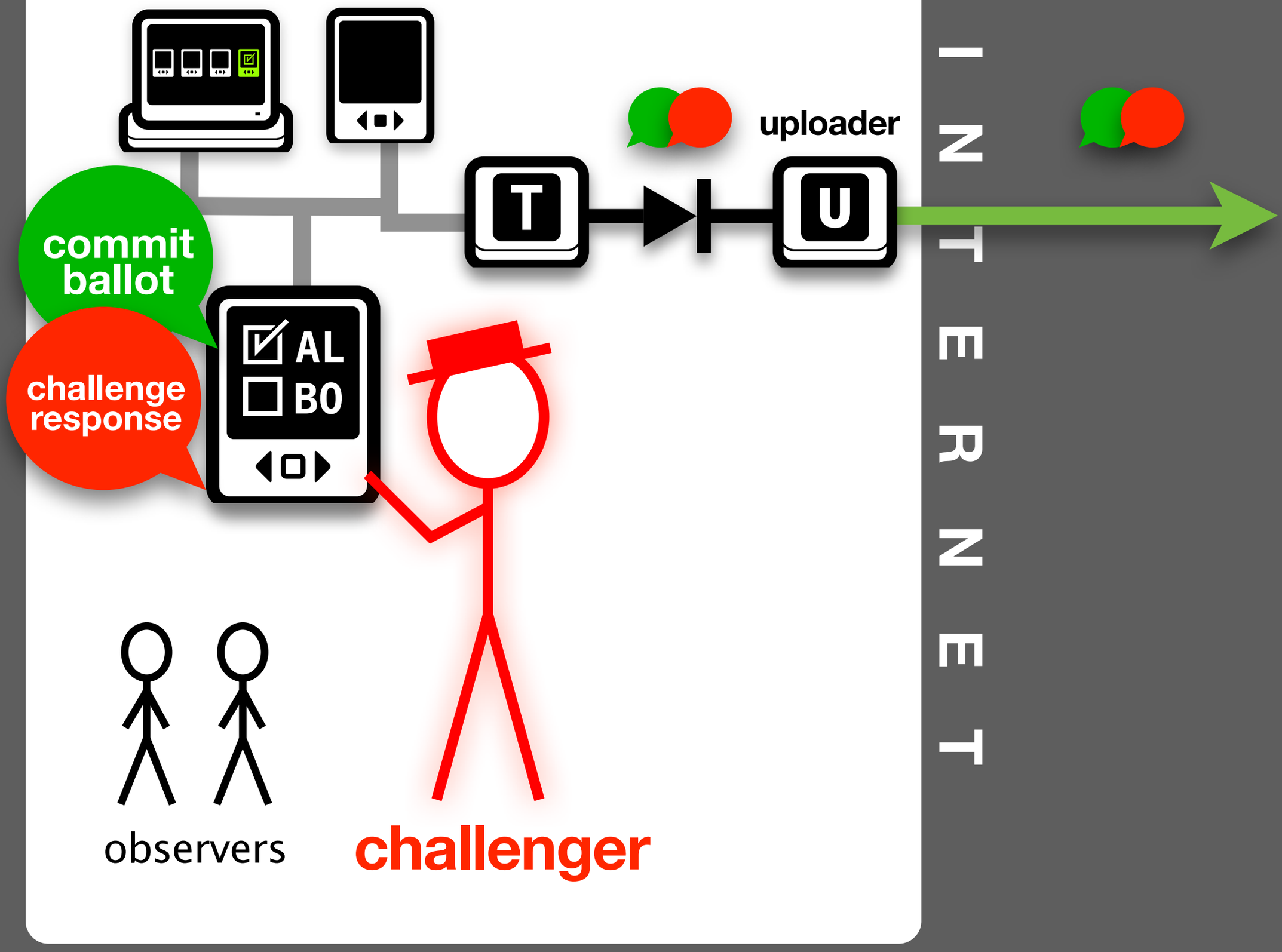


polling place



polling place

challenge center

INTERNET

uploader

T

U

commit ballot

challenge response

AL

BO

observers

challenger

polling place

challenge center

INTERNET

uploader

T

U

commit ballot

challenge response

AL

BO

challenge verification results

observers

challenger

polling place

challenge center

INTERNET

commit ballot

challenge response

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ALICE

challenge verification results

observers

challenger